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LETTER TO THE EDITOR

Kadomtsev–Petviashvili equation with a source and its soliton solutions

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Abstract. A new class of coupled soliton equations in $2+1$ dimensions is proposed. They describe the interactions of a wavepacket of short waves with a single long wave on the xy plane. The N-soliton solutions of the equations are obtained explicitly. The proposed system of equations is also shown to include, as special cases, various known physical equations, indicating that the system is relevant to describe physical systems and may have wide applications in hydrodynamics and plasma physics, etc.

In this letter, we propose the following new class of soliton equations in two spatial and one temporal i.e. $(2+1)$ dimensions

$$(u_t+6uu_x+u_{xxx})_x-3u_{yy}=-\int_{-\infty}^{\infty}\mathrm{d}k\,\nu(k,t)(|\phi|^2)_{xx} \tag{1}$$

$$\mathrm{i}\phi_y=\phi_{xx}+u\phi \tag{2}$$

where $u=u(x,y,t)$ and $\phi=\phi(x,y,t;k)$ are assumed to be real and complex functions respectively; k is a real parameter and $\nu(k,t)$ is a given real function. The subscripts appended to u and ϕ denote partial differentiation. Physically, the above system of equations describes the interaction of a wavepacket of short waves ϕ with a single long wave u on the xy plane. The integral over k in (1) represents the effect of a wavepacket. Since when $\phi=0$, (1) and (2) reduce to the Kadomtsev–Petviashvili (KP) equation, we call it the KP equation with a source. Our system reproduces various known physical systems under suitable conditions. For instance, if we consider a single short wave instead of a wavepacket, i.e. $\nu(k,t)=\kappa\delta(k-k_0)$, we recover the Mel'nikov equation [1-3]. Also neglecting the y-dependence, it reduces to the Korteweg-de Vries (KDV) equation with a source [4-6]. These facts show that the system (1) and (2) is relevant to describe physical systems and may have wide applications in various fields of hydrodynamics and plasma physics, etc.

The purpose of the present letter is to solve (1) and (2) under the following boundary conditions:

$$u\to 0 \qquad \text{as} \qquad |x|\to\infty \qquad \text{with fixed } y \tag{3}$$

$$\phi\to\phi_0(k,t)\exp[\mathrm{i}(kx+k^2y)] \qquad \text{as} \qquad x\to-\infty \tag{4}$$

where ϕ_0 is a given function of k and t. The boundary condition (4) implies that the source is situated at infinity. Although one can consider more general initial-boundary value problems, this difficult subject will be treated elsewhere and we shall be concerned

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with constructing soliton solutions. The method of solution employed here is the so-called bilinear transformation method [7, 8].

Let us now seek solutions. For the purpose, we introduce the dependent variable transformations

$$u = 2(\ln f_N)_{xx} \tag{5}$$

$$\phi = \phi_0 \exp[\mathrm{i}(kx + k^2 y)] g_N / f_N \tag{6}$$

where $f_N(g_N)$ is a real (complex) function. Substitution of (5) and (6) into (1) and (2) yields the following bilinear equations for f_N and g_N:

$$(D_t D_x - 3D_y^2 + D_x^4) f_N \cdot f_N = -\int_{-\infty}^{\infty} \mathrm{d}k\, \nu |\phi_0|^2 (|g_N|^2 - f_N^2) \tag{7}$$

$$\mathrm{i} D_y g_N \cdot f_N = (D_x^2 + 2\mathrm{i}kD_x) g_N \cdot f_N. \tag{8}$$

Here, the bilinear operators D_t, D_x and D_y are defined by

$$D_t^l D_x^m D_y^n g \cdot f = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^l \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n g(t, x, y) f(t', x', y') \bigg|_{\substack{t'=t \\ x'=x \\ y'=y}}. \tag{9}$$

It should be emphasized that (7) is a new type of bilinear equation because of the presence of the k integral on the right-hand side of (7) and this makes the analysis more complicated than that of the usual system of bilinear equations. Nevertheless, following the procedure of the bilinear transformation method [7, 8], we have found the N-soliton solutions of (7) and (8) as follows:

$$f_N = \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j \eta_j + \sum_{1 \leq j < k \leq N} \mu_j \mu_k A_{jk} \right) \tag{10}$$

$$g_N = \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j (\eta_j + \mathrm{i}\alpha_j) + \sum_{1 \leq j < k \leq N} \mu_j \mu_k A_{jk} \right) \tag{11}$$

with

$$\eta_j = (p_j + p_j^*)x - \mathrm{i}(p_j^2 - p_j^{*2})y - 4(p_j^3 + p_j^{*3})t + \xi_j(t) + \delta_j \qquad (j = 1, 2, \ldots, N) \tag{12}$$

$$\xi_j(t) = \frac{1}{2}(p_j + p_j^*) \int_0^t \mathrm{d}t' \int_{-\infty}^{\infty} \frac{\nu(k, t') |\phi_0(k, t')|^2}{k^2 + \mathrm{i}(p_j - p_j^*)k + p_j^* p_j} \mathrm{d}k \qquad (j = 1, 2, \ldots, N) \tag{13}$$

$$\exp(A_{jk}) = \frac{(p_j - p_k)(p_j^* - p_k^*)}{(p_j + p_k^*)(p_j^* + p_k)} \qquad (j, k = 1, 2, \ldots, N; j \neq k) \tag{14}$$

$$\exp(\mathrm{i}\alpha_j) = \frac{k + \mathrm{i}p_j}{k - \mathrm{i}p_j^*} \qquad (j = 1, 2, \ldots, N). \tag{15}$$

Here, p_j are complex parameters whose real parts are positive, δ_j are real phase constants, the asterisk denotes the complex conjugate and $\Sigma_{\mu=0,1}$ indicates the summation over all possible combinations of $\mu_1 = 0, 1$, $\mu_2 = 0, 1, \ldots, \mu_N = 0, 1$.

Explicitly for $N=1$, (10) and (11) are written as

$$f_1 = 1 + e^{\eta_1} \tag{16}$$

$$g_1 = 1 + e(\eta_1 + i\alpha_1). \tag{17}$$

The expressions for u and ϕ are given by using (5) and (6) in the form

$$u = \tfrac{1}{2}(p_1 + p_1^*)^2 \operatorname{sech}^2(\eta_1/2) \tag{18}$$

$$\phi = \phi_0 \exp[i(kx + k^2 y)] \frac{1 + \exp(\eta_1 + i\alpha_1)}{1 + e^{\eta_1}}. \tag{19}$$

We note here that the functional form (18) is the same as that of the one-soliton solution [9] of the KP equation, the only difference being the position of the soliton. The solution (19) represents a soliton of dark type. The interaction of solitons can easily be investigated by using (10)-(15) and it is found that the N-soliton solution evolves asymptotically as $t \to \pm\infty$ into N plane-wave-like solitons. However, the dynamics of solitons change from those of the usual KP solitons due to the coupling between u and ϕ. The detailed description of the interaction process will be reported in a subsequent paper.

Next, we show that various known equations arise from our system by the appropriate choices of parameters included in problems under consideration and discuss their soliton solutions. The four interesting cases are considered below.

(i) Mel'nikov equation.

We put $\nu(k, t) = \kappa\delta(k - k_0)$ in (1), which corresponds to take account of a monochromatic short wave instead of a wavepacket. Performing the integral over k yields the Mel'nikov equation for which various types of solutions have been presented [1-3, 10].

(ii) KDV equation with a source.

This system arises by introducing new variables $\tilde{\phi}$ by

$$\phi = \exp(ik^2 y_{\tilde{\phi}}) \tag{20}$$

and neglecting the y-dependence. Then, (1) and (2) reduce, after integrating (1) once with respect to x, to the system of equations

$$u_t + 6uu_x + u_{xxx} = -\int_{-\infty}^{\infty} dk\, \nu(k, t)(|\tilde{\phi}|^2)_x \tag{21}$$

$$\tilde{\phi}_{xx} + (u + k^2)\tilde{\phi} = 0 \tag{22}$$

which is nothing but the KDV equation with a source [4-6]. The N-soliton solutions readily follow from our solutions by taking p_j real, i.e., $p_j = p_j^* (j = 1, 2, \ldots, N)$. The expressions corresponding to (12)-(15) read as follows:

$$\eta_j = 2p_j x - 8p_j^3 t + \xi_j(t) + \delta_j \qquad (j = 1, 2, \ldots, N) \tag{23}$$

$$\xi_j(t) = p_j \int_0^t dt' \int_{-\infty}^{\infty} \frac{\nu(k, t')|\phi_0(k, t')|^2}{k^2 + p_j^2} dk \qquad (j = 1, 2, \ldots, N) \tag{24}$$

$$\exp(A_{jk}) = \left(\frac{p_j - p_k}{p_j + p_k}\right)^2 \qquad (j, k = 1, 2, \ldots, N; j \neq k) \tag{25}$$

$$\exp(i\alpha_j) = \frac{k + ip_j}{k - ip_j} \qquad (j = 1, 2, \ldots, N). \tag{26}$$

(iii) Boussinesq equation with a source.

If we introduce new independent variables ξ, η and τ by

$$\xi = x + t \qquad \eta = y \qquad \tau = t \tag{27}$$

and neglect the τ-dependence, (1) and (2) are transformed into the form

$$u_{\xi\xi} + (6uu_\xi)_\xi + u_{\xi\xi\xi\xi} - 3u_{\eta\eta} = -\int_{-\infty}^{\infty} \mathrm{d}k\, \nu(k)(|\phi|^2)_{\xi\xi} \tag{28}$$

$$\mathrm{i}\phi_\eta = \phi_{\xi\xi} + u\phi. \tag{29}$$

Regarding η and ξ as the time and space variables respectively, one arrives at the Boussinesq equation with a source. Since the N-soliton solutions should not depend on τ, we require that the coefficient of τ in η_j $(j = 1, 2, \ldots, N)$ vanishes identically, which is equivalent to imposing the conditions

$$4(p_j^2 - p_j^* p_j + p_j^{*2}) + 1 - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\nu(k)|\phi_0(k)|^2}{k^2 + \mathrm{i}(p_j - p_j^*)k + p_j^* p_j}\, \mathrm{d}k = 0 \qquad (j = 1, 2, \ldots, N). \tag{30}$$

The real part of p_j is then related to its imaginary part, yielding the N independent amplitude parameter. It is interesting to remark that the system with $\nu(k) = \kappa\delta(k - k_0)$ coincides with a system of equations proposed by Hase and Satsuma [11].

(iv) Generalized Karpman equations.

Introduce new scaled variables $\tilde{x}$, $\tilde{u}$, $\tilde{k}$ and $\tilde{p}_j$ by

$$x = \varepsilon^{-1}\tilde{x} \qquad u = \varepsilon^2 \tilde{u} \qquad k = \varepsilon \tilde{k} \qquad p_j = p_j^* = \varepsilon \tilde{p}_j \qquad (j = 1, 2, \ldots, N) \tag{31}$$

together with (20), substituting these into (1) and (2) and neglecting the y-dependence, we obtain in the limit of $\varepsilon \to 0$

$$\tilde{u}_t = -\int_{-\infty}^{\infty} \mathrm{d}\tilde{k}\, \nu(\tilde{k}, t)(|\tilde{\phi}|^2)_{\tilde{x}} \tag{32}$$

$$\tilde{\phi}_{\tilde{x}\tilde{x}} + (\tilde{u} + \tilde{k}^2)\tilde{\phi} = 0 \tag{33}$$

which are the generalized Karpman equations introduced by Kaup [12, 13] to describe caviton formation in a plasma. The $\tilde{k}$ integral in (32) represents the cumulative pondermotive effect caused by a spectrum of incoherent and/or turbulent sources. The n-soliton solutions of (32) and (33), when expressed in terms of the new variables (31), take the same forms as those for the KDV equation with a source (see (23)-(26)) except that the phase factors η_j are replaced by $\eta_j = 2\tilde{p}_j\tilde{x} + \xi_j(t) + \delta_j$. These solutions reproduce those obtained by Kaup using the inverse scattering method [12, 13].

In this letter, we have proposed a system of equations (1) and (2), and found explicit N-soliton solutions under a vanishing boundary condition for a long wave, and a non-vanishing one for a wavepacket of short waves. It has been shown that our system of equations exhibits a new aspect of soliton interactions in 2+1 dimensions and it also includes various physical systems as special cases. From the mathematical point of view, on the other hand, we can generalize (1) such that the left-hand side of (1) becomes the first member of the KP hierarchy [14]. The corresponding bilinear equation would take the form

$$[D_t D_x + P_n(D_x, D_y, D_z, \ldots)] f_N \cdot f_N = -\int_{-\infty}^{\infty} \mathrm{d}k\, \nu |\phi_0|^2 (|g_N|^2 - f_N^2) \qquad (n = 1, 2, \ldots.) \tag{34}$$

where $P_n(x, y, z, \ldots)$ are certain polynomials of $x, y, z, \ldots$, the first two of which are given by $P_1 = x^4 - 3y^2$ and $P_2 = x^3y + 2yz$. One can construct N-soliton solutions of (8) and (34) on the basis of the general theory of the KP equation [14]. The result is quite simple. The functional forms of N-soliton solutions are the same as those of (1) and (2) except that the phase factors (12) are slightly modified. Hence, the N-soliton solutions of corresponding hierarchies of equations metnioned in (i)-(iv) also follow from ours by appropriate reductions.

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References

[1] Mel'nikov V K 1983 *Lett. Math. Phys.* **7** 129
[2] Mel'nikov V K 1986 *Phys. Lett.* **118A** 22
[3] Mel'nikov V K 1987 *Commun. Math. Phys.* **112** 639; 1989 *Commun. Math. Phys.* **120** 451
[4] Mel'nikov V K 1988 *Phys. Lett.* **133A** 494
[5] Mel'nikov V K 1990 *Inverse Problems* **6** 233
[6] Leon J and Latifi A 1990 *J. Phys. A: Math. Gen.* **23** 1385
[7] Hirota R 1971 *Phys. Rev. Lett.* **27** 1192
[8] Matsuno Y 1984 *Bilinear Transformation Method* (New York: Academic)
[9] Satsuma J 1976 *J. Phys. Soc. Japan* **40** 286
[10] Hase Y, Hirota R, Ohta Y and Satsuma J 1989 *J. Phys. Soc. Japan* **58** 2713
[11] Hase Y and Satsuma J 1988 *J. Phys. Soc. Japan* **57** 679
[12] Kaup D J 1987 *Phys. Rev. Lett.* **59** 2063
[13] Kaup D J 1988 *Phys. Fluids* **31** 1465
[14] Jimbo M and Miwa T 1983 *Publ. RIMS* **19** 943